Understanding the Focus of a Parabola and Solving Example Problems

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Why Are We Interested in Locating the Focus of a Parabola?

The parabola is an important geometric shape, not just in mathematics but also in various fields like physics, engineering, and astronomy. Parabolic curves have unique properties that allow them to focus or direct waves - whether they are light waves, sound waves, or even gravitational effects. This makes the focus of a parabola crucial in numerous real-world applications.

For instance:

- **In satellite dishes and telescopes**, signals (or light) are reflected toward the focus, concentrating them to a point for clearer reception or observation.
- **In flashlights and car headlights**, a parabolic reflector directs light outward from a bulb placed at the focus.
- **In physics**, a particle under the influence of gravity follows a parabolic trajectory, while in bridge design, the cables or arches of suspension and arch bridges follow parabolic curves to efficiently distribute weight.

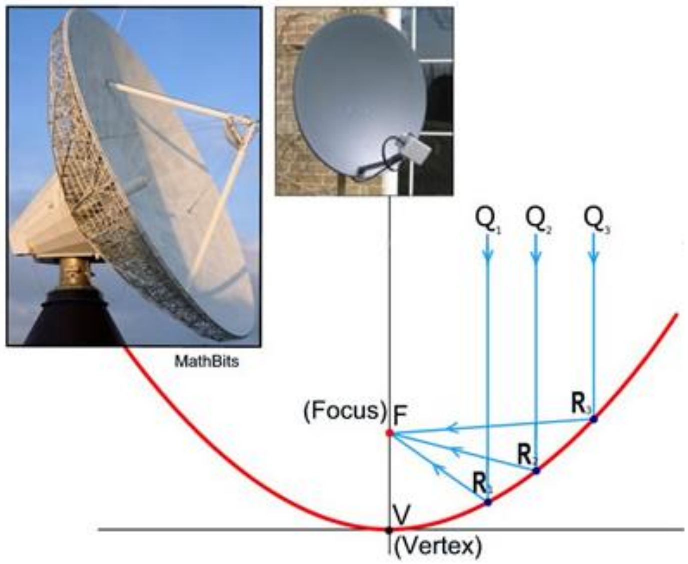
To fully harness the power of these parabolic curves, it's essential to derive and solve the equation of the parabola, particularly in terms of locating its focus.

Applications of Parabolic Curves

1. Parabolic Dish

A parabolic dish, or reflector, is a curved surface with a cross-sectional shape of a parabola. This surface is used to direct sound or light waves to a specific point. Any sound waves hitting the surface parallel to its axis of symmetry are reflected to the dish's focus, concentrating energy.

Real-World Example: Radio telescopes and satellite dishes place a receiver at the focus of the parabolic reflector to capture a stronger signal. Similarly, the parabolic reflector in a flashlight concentrates light emitted from a bulb into a beam.



2. Parabolas in Physics

In physics, a well-known example is the parabolic trajectory of objects under the influence of gravity, a concept first discovered by Galileo. Without air resistance, such a trajectory approximates a perfect parabola.

Real-World Example: The parabolic trajectory can even simulate zero-gravity environments, as shown in the case of Stephen Hawking experiencing a parabolic flight, which creates brief moments of weightlessness.

3. Parabolas in Bridge Design

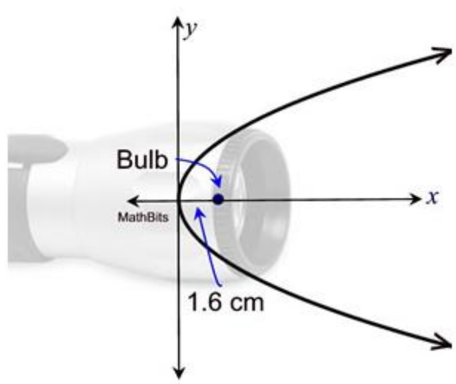
- **Suspension bridges** use parabolic curves to describe the shape of the cables. The weight of the bridge is efficiently distributed across these curves.

- **Arch bridges** use parabolic curves to transfer the weight of the bridge into the ground through the supporting arch.

Solving Example Problems

Example 1: The Parabolic Reflector in a Flashlight

Problem Statement:
The parabolic reflector in a flashlight focuses light from a bulb placed 1.6 cm from the vertex. We need to find the equation of the parabola with the vertex at the origin and focus on the positive x-axis.



Solution:

1. **Key Information:**
    - The vertex of the parabola is at $(0,0)$.
    - The focus is 1.6 cm from the vertex along the x-axis, at $(1.6,0)$.
2. **Standard Equation of a Parabola:**
   The equation of a parabola with a horizontal axis and vertex at the origin is:

$$y^2 = 4px$$

Where p is the distance from the vertex to the focus. Here, $p = 1.6$.

3. **Substitute p into the Equation:**

$$y^2 = 4(1.6)x$$

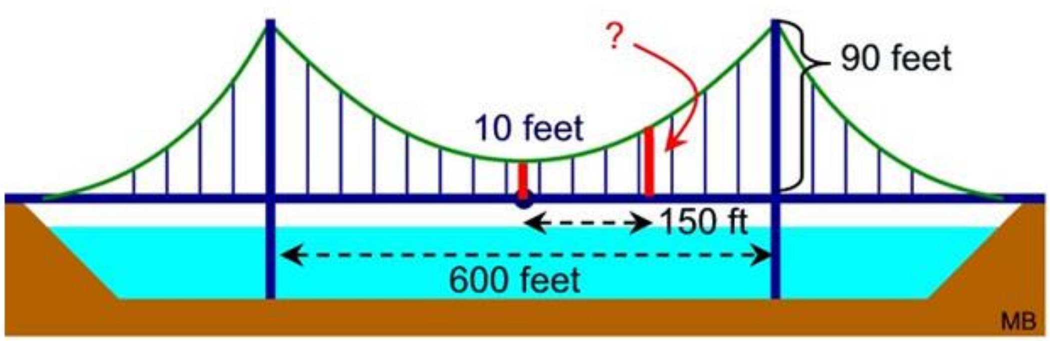
$$y^2 = 6.4x$$

Thus, the equation of the parabolic reflector is:

$$y^2 = 6.4x$$

Example 2: The Suspension Bridge

Problem Statement:
The cables of a suspension bridge are in the shape of a parabola. The pillars supporting the cables are 600 feet apart and rise 90 feet above the road. The lowest point of the cable is 10 feet above the road. We need to find the height of the cable at a point 150 feet from the center.



Solution:

1. **Setting the Coordinate System:**

- Place the x-axis along the road and the y-axis at the midpoint of the bridge. The vertex of the parabola is at the lowest point of the cable, which is 10 feet above the road, so the vertex is at $(0,10)$.
- The cables are supported by pillars that are 600 feet apart, with the height of the cable being 90 feet at the pillars, so the point $(300,90)$ lies on the parabola.

2. **Standard Equation of a Parabola:**
   The equation of the parabola is:

$$y = a(x-h)^2 + k$$

Since the vertex is at $(0,10)$, the equation becomes:

$$y = ax^2 + 10$$

3. **Finding a:**
   Substituting the point $(300,90)$ into the equation:

$$90 = a(300)^2 + 10$$

$$80 = a(90000)$$

$$a = \frac{80}{90000} = \frac{2}{2250}$$

4. **Full Equation:**
   The equation of the parabola is:

$$y = \frac{2}{2250}x^2 + 10$$

5. **Finding the Height at 150 Feet:**
   Substitute $x = 150$ into the equation:

$$y = \frac{2}{2250}(150)^2 + 10$$

$$y = \frac{2}{2250}(22500) + 10$$

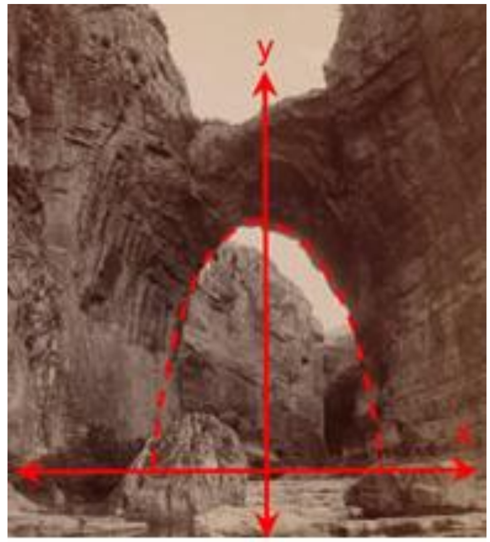
$$y = 20 + 10 = 30$$

Thus, the height of the cable 150 feet from the center is 30 feet.

Example 3: The Natural Arch

Problem Statement:

A natural stone bridge has an arch in the shape of a parabola. The height of the arch is 50 feet, and the width at the base is 30 feet. We need to write an equation to model this arch.



Solution:

1. **Setting the Coordinate System:**
   - The base of the arch is along the x-axis, and the highest point of the arch is at $(0,50)$, so the vertex is $(0,50)$.
   - The width at ground level is 30 feet, so the x-intercepts are $(15,0)$ and $(-15,0)$.
2. **Standard Equation of a Parabola:**
   The equation of the parabola is:

$$(x-h)^2 = 4p(y-k)$$

With the vertex at $(0,50)$, the equation simplifies to:

$$x^2 = 4p(y-50)$$

3. **Finding p:**
   Substituting the point $(15,0)$ into the equation:

$$15^2 = 4p(0-50)$$

$$225 = -200p$$

$$p = -\frac{9}{8}$$

4. **Full Equation:**
   The equation of the arch is:

$$x^2 = 4\left(-\frac{9}{8}\right)(y - 50)$$

$$x^2 = -\frac{9}{2}(y - 50)$$

Thus, the equation modeling the arch is:

$$x^2 = -\frac{9}{2}(y - 50)$$

Conclusion

The focus of a parabola plays a critical role in various fields by helping to direct or collect waves (light, sound, etc.). By solving these example problems, we see how parabolic curves are used to concentrate light in flashlights, design suspension bridges, and model natural arches. Understanding how to locate the focus and derive the equation of a parabola is essential for applying these concepts in real-world situations.