

- ✓ 3. [Maximum mark: 16]

Tiffany wants to buy a house for a price of 285 000 US Dollars (USD). She goes to a bank to get a loan to buy the house. To be eligible for the loan, Tiffany must make an initial down payment equal to 15% of the price of the house.

The bank offers her a 30-year loan for the remaining balance, with a 4% nominal interest rate per annum, compounded monthly. Tiffany will pay the loan in fixed payments at the end of each month.

- (a) (i) Find the original amount of the loan after the down payment is paid.  
Give the exact answer.
- (ii) Calculate Tiffany's monthly payment for this loan, to two decimal places. [5]
- (b) Using your answer from part (a)(ii), calculate the total amount Tiffany will pay over the life of the loan, to the nearest dollar. Do **not** include the initial down payment. [2]

Tiffany would like to repay the loan faster and increases her payments such that she pays 1300 USD each month.

- (c) Find the total number of monthly payments she will need to make to pay off the loan. [2]  
This strategy will result in Tiffany's final payment being less than 1300 USD.

- (d) Determine the amount of Tiffany's final payment, to two decimal places. [4]
- (e) Hence, determine the total amount Tiffany will save, to the nearest dollar, by making the higher monthly payments. [3]

however the 291.6725682 → 292 months, the 125th payment won't be

- (d) Determine the amount of Tiffany's final payment, to two decimal places.

- (e) Hence, determine the total amount Tiffany will save, to the nearest dollar, by making the higher monthly payments.

✓ (d)  $0.6725682 \times 1,300 \$ =$

(this is the remainder for the 292nd month)

$874.34 \$$  (874.3384)

(e) Amount saved =  $360 \cdot 1,156.54 - 292 \times 1,300$   
A. Saved =  $37,180 \$ - 874.34$

$$\textcircled{3} \textcircled{i} \text{ Loan Amount} = \text{House Price} - \text{Down Payment} \\ = 285,000 - 0.15 \cdot 285,000 = 242,250.$$

$\textcircled{ii} \text{ PMT} = ?$

Compounding Periods:  $N = 30 \text{ years} \cdot 12 = 360.$

Interest:  $I = 4\%$

Payment Per Year:  $P/Y = 12$

Compounds per year (monthly):  $C/Y = 12$

$FV = 0$  (future value). }  $\Rightarrow \text{PMT} = 1156.54$  \$

$PV = 242,250$

$\textcircled{6} \text{ Total Amount Tiffany Pays:}$

$$360 \text{ months} \times 1,156.54 \$ = 416,354$$

$\textcircled{c} \text{ New PMT} = -1,300 \$ \text{ (USD)}$

and  $P.V. = 242,250$

$N = ? \Rightarrow 292 \text{ months}$

$$\textcircled{d} 0.6725682 \times 1,300 \$ = 874.34 \$$$

final month is not full.

291 full months

savings  
 $\textcircled{e} 360 \cdot 1,156.54 -$

$$291.67 \times 1,300 =$$

37,183 \$

(USD)

Maths A.I. HL Paper II (October) x Mats A.I. HL Paper I (October 20) x +

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3. [Maximum mark: 16]

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(ii) Calculate Tiffany's monthly payment for this loan, to two decimal places. [5]

(b) Using your answer from part (a)(ii), calculate the total amount Tiffany will pay over the life of the loan, to the nearest dollar. Do not include the initial down payment. [2]

Tiffany would like to repay the loan faster and increases her payments such that she pays 1300 USD each month.

(c) Find the total number of monthly payments she will need to make to pay off the loan. [2]

*292 (rounded up)*

This strategy will result in Tiffany's final payment being less than 1300 USD.

(d) Determine the amount of Tiffany's final payment, to two decimal places. [4]

(e) Hence, determine the total amount Tiffany will save, to the nearest dollar, by making the higher monthly payments. [3]

$\$ \text{ (USD)}$

**IQR is basically the "middle" 50%**

HA

2. [Maximum mark: 18]

The heights,  $h$ , of 200 university students are recorded in the following table.

Height (cm)	Frequency
$140 \leq h < 160$	11
$160 \leq h < 170$	51
$170 \leq h < 180$	68
$180 \leq h < 190$	47
$190 \leq h < 210$	23

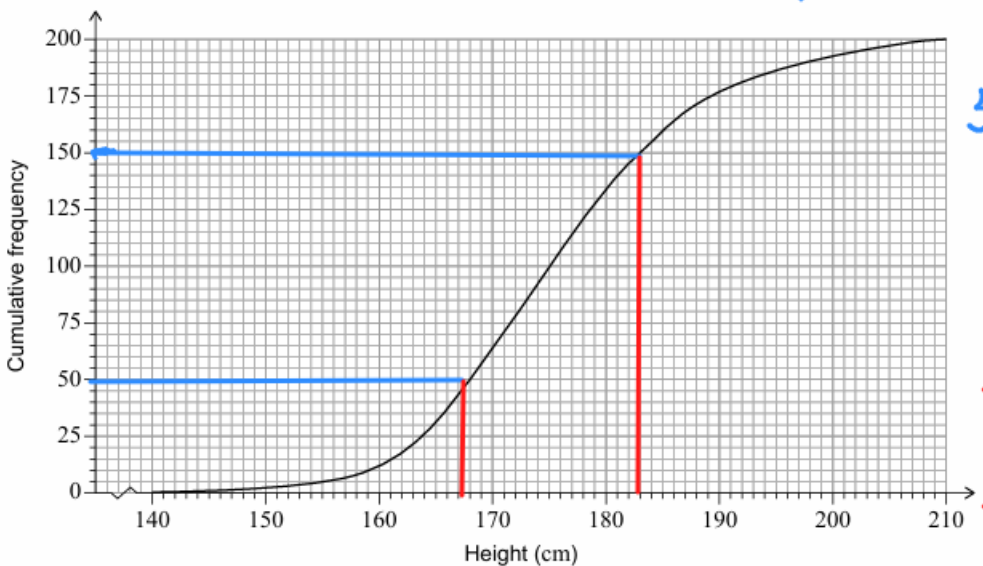
$$\bar{X} = \frac{\sum f_i x_i}{\sum f_i} = \frac{x_1 \cdot f_1 + x_2 \cdot f_2 + x_3 \cdot f_3 + \dots}{f_1 + f_2 + \dots + f_n}$$

$$\bar{X} = \frac{1,650 + 8,415 + 11,900 + 8,695 + 4600}{200} = 176.3 \text{ cm}$$

- (a) (i) Write down the mid-interval value of  $140 \leq h < 160$ .  
 (ii) Calculate an estimate of the mean height of the 200 students.

$= 150$  is the mid interval  
 [3]

This table is used to create the following cumulative frequency graph.



$150 \rightarrow 75\% (Q_3)$   
 $50 \rightarrow 25\% (Q_1)$

$Q_3 = 182.5$   
 $Q_1 = 167.5$   
 $IQR = Q_3 - Q_1$   
 $IQR = 182.5 - 167.5$   
 $IQR = 15 \text{ cm}$

(b) Use the cumulative frequency curve to estimate the interquartile range.

Laszlo is a student in the data set and his height is 204 cm.

(c) Use your answer to part (b) to estimate whether Laszlo's height is an outlier for this data. Justify your answer.

$\odot$  Upper Bound:  $Q_3 + 1.5IQR = 205 \text{ cm} > 204 \text{ cm}$  Laszlo's height so not an outlier

(Question 2 continued)

It is believed that the heights of university students follow a normal distribution with mean 176 cm and standard deviation 13.5 cm.

It is decided to perform a  $\chi^2$  goodness of fit test on the data to determine whether this sample of 200 students could have plausibly been drawn from an underlying distribution  $N(176, 13.5^2)$ .

(d) Write down the null and the alternative hypotheses for the test.

As part of the test, the following table is created.

Height of student (cm)	Observed frequency	Expected frequency
$h < 160$	11	23.6
$160 \leq h < 170$	51	42.1
$170 \leq h < 180$	68	$a$
$180 \leq h < 190$	47	46.7
$190 \leq h$	23	$b$

- (e) (i) Find the value of  $a$  and the value of  $b$ .  
 (ii) Hence, perform the test to a 5% significance level, clearly stating the conclusion in context.

$\odot$   $H_0$ : The heights of the university students follow a normal distribution with  $N(\mu = 176 \text{ cm}, \sigma^2 = 13.5^2)$ .  
 $H_1$ : The heights of the university students do not follow a normal distribution.

**IQR is basically the "middle" 50%**

HA

(e)  $\left. \begin{array}{l} \text{Lower: } 190 \\ \text{Upper: } 1 \times 10^{99} \end{array} \right\} \text{Normal c.d.f.}$   
 $m = 176 / \sigma = 13.5$

$p = 0.149 (0.14985929)$

(ii)  $H_0$ : Uni heights follow N.D.  $n - 1 = 5 - 1 = 4$ . clusters = classes

" " " " " " of students do not follow N.D.

Talking: Mr Alex

GOF test:  $\chi = 12.1$

$p = 0.0165 (0.01646934) < 0.05$

$df = 4$   
do not

So therefore we accept the Null hypothesis, the heights of the university students follow a normal distribution trend. do not seem.

(Question 2 continued)

It is believed that the heights of university students follow a normal distribution with mean 176 cm and standard deviation 13.5 cm.

It is decided to perform a  $\chi^2$  goodness of fit test on the data to determine whether this sample of 200 students could have plausibly been drawn from an underlying distribution  $N(176, 13.5^2)$ .

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$180 \leq h < 190$	47	46.7
$190 \leq h$	23	$b$

Expectation =  $p \cdot \text{population} = 200 \cdot 0.288 = 57.6$

unbiased (it is what it is)

= to follow

p-value Normal distribution for this probability

(e) (i) Find the value of  $a$  and the value of  $b$ . [4]

(ii) Hence, perform the test to a 5% significance level, clearly stating the conclusion in context. [8]

verification purposes

$23.6 + 42.1 + a + 46.7 + b = 200$

- (i) List 1  
11  
51  
68  
47  
23

$b = 200 - 23.6 - 42.1 - 46.7 - 57.6$

$b = 30.0$

so not

$$(e^{-0.07x})' = e^{-0.07x} \cdot (-0.07) = -0.07 \cdot e^{-0.07x}$$

chain rule  $(f(g(x)))' = f'(g(x)) \cdot g'(x)$

4. [Maximum mark: 12]

A plane takes off from a horizontal runway. Let point O be the point where the plane begins to leave the runway and  $x$  be the horizontal distance, in km, of the plane from O. The function  $h$  models the vertical height, in km, of the nose of the plane from the horizontal runway, and is defined by

$$h(x) = \frac{10}{1 + 150e^{-0.07x}} - 0.06, \quad x \geq 0.$$

diagram not to scale

(a) (i) Find  $h(0)$ . [2]  
 (ii) Interpret this value in terms of the context. [2]

(b) (i) Find the horizontal asymptote of the graph of  $y = h(x)$ . [2]  
 (ii) Interpret this value in terms of the context. [2]

(c) Find  $h'(x)$  in terms of  $x$ . [4]

A safety regulation recommends that  $h'(x)$  never exceed 0.2. [2]

(d) Given that this plane flies a distance of at least 200 km horizontally from point O, determine whether the plane is following this safety regulation. [4]

**Even after an infinite (very high) amount of km covered horizontally (horizontal distance), the nose of the of the plane cannot practically exceed 10km in terms of height.**

4. [Maximum mark: 12]

A plane takes off from a horizontal runway. Let point O be the point where the plane begins to leave the runway and  $x$  be the horizontal distance, in km, of the plane from O. The function  $h$  models the vertical height, in km, of the nose of the plane from the horizontal runway, and is defined by

$$h'(x) = \left( \frac{10}{1 + 150 \cdot e^{-0.07x}} - 0.06 \right)' \Rightarrow$$

$$h'(x) = \left( \frac{10}{1 + 150 \cdot e^{-0.07x}} \right)' - (0.06)' = \frac{10 \cdot (-1) \cdot (-0.07 \cdot e^{-0.07x})}{(1 + 150 \cdot e^{-0.07x})^2} - 0 = \frac{0.7 \cdot 10 \cdot e^{-0.07x}}{(1 + 150 \cdot e^{-0.07x})^2}$$

$$h'(x) = \frac{-10 \cdot (-0.07 \cdot e^{-0.07x})}{(1 + 150 \cdot e^{-0.07x})^2} = \frac{0.7 \cdot 10 \cdot e^{-0.07x}}{(1 + 150 \cdot e^{-0.07x})^2}$$

so  $y = \frac{10}{1 + 150 \cdot 0} - 0.06 = 10 - 0.06 = 9.94 \text{ km}$  horizontal asymptote.

so as  $x \rightarrow +\infty \rightarrow e^{-0.07 \cdot (+\infty)} = \frac{1}{e^{+\infty}} \approx 0$ .

(i) When the plane hasn't started its journey ( $x = 0 \text{ km}$ ) so it's on the runway, it is the initial height of the nose of the plane.

(b) since  $x \geq 0$  we will investigate what happens/so as  $x \rightarrow +\infty$  (which in real life means a very high amount of km).

A safety regulation recommends that  $h'(x)$  never exceed 0.2.

- (d) Given that this plane flies a distance of at least 200 km horizontally from point O, determine whether the plane is following this safety regulation.

[4]

Explain everything.

≠ solve  $h'(x) = 0 \Rightarrow$

**Note:** Award **A1** for correct numerator and **A1** for correct denominator.

[4 marks]

{

- (d) evidence of a graph of  $h'(x)$   
maximum at  $x = 71.6$  ( $= 71.58051\dots$ )  
 $h'(71.58051\dots) = 0.175$   
maximum gradient is less than 0.2  
and hence the regulation is being followed

(M1)  
(A1)  
A1

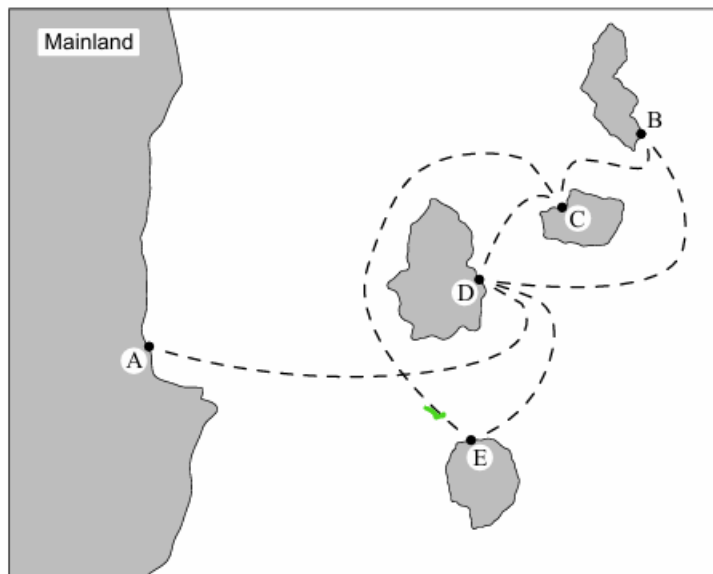
A1

[4 marks]

[Total 12 marks]

5. [Maximum mark: 18]

The following diagram is a map of a group of four islands and the closest mainland. Travel from the mainland and between the islands is by boat. The scheduled boat routes between the ports A, B, C, D and E are shown as dotted lines on the map.



Let the undirected graph  $G$  represent the boat routes between the ports A, B, C, D and E.

(a) Draw graph  $G$ . ✓

(b) Graph  $G$  can be represented by an adjacency matrix  $P$ , where the rows and columns represent the ports in alphabetical order.

✓ (i) Given that  $P^3 = \begin{pmatrix} 0 & 1 & 2 & 4 & 1 \\ 1 & 2 & 5 & a & 2 \\ 2 & 5 & 4 & 6 & 5 \\ 4 & a & 6 & 4 & 6 \\ 1 & 2 & 5 & 6 & 2 \end{pmatrix}$ , find the value of  $a$ .

✓ (ii) Hence, write down the number of different ways that someone could start at port B and end at port C, using three boat route journeys.

✓ (c) Find a possible Eulerian trail in  $G$ , starting at port A.

(This question continues on the following page)



(b) 
$$P = \begin{matrix} & \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} & \begin{bmatrix} 0 & 1 & 2 & 4 & 1 \\ 1 & 2 & 5 & a & 2 \\ 2 & 5 & 4 & 6 & 5 \\ 4 & a & 6 & 4 & 6 \\ 1 & 2 & 5 & 6 & 2 \end{bmatrix} \end{matrix} \Rightarrow P^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & 1 & 2 & 4 & 1 \\ 1 & 2 & 5 & a & 2 \\ 2 & 5 & 4 & 6 & 5 \\ 4 & a & 6 & 4 & 6 \\ 1 & 2 & 5 & 6 & 2 \end{bmatrix}$$
  
 $a = 6$

1st. B - D - B - C } Based on the adjacency matrix,  
 2nd. B - C - D - C } there are 5 different for 3-boat  
 3rd. B - D - E - C } route journeys.  
 4th. B - C - E - C }  
 5th.

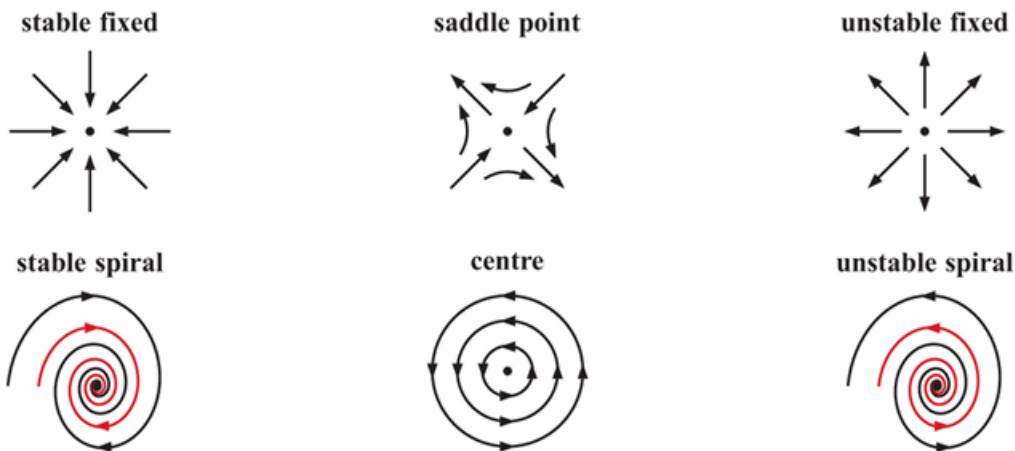
(c) Eulerian Trail (you don't have to get back to A), just visit every edge: A - D - E - C - B or A - D - B - C - E

[3]

[2]

## EQUILIBRIUM POINTS

We can describe an equilibrium point according to the behaviour of the phase portrait around it.



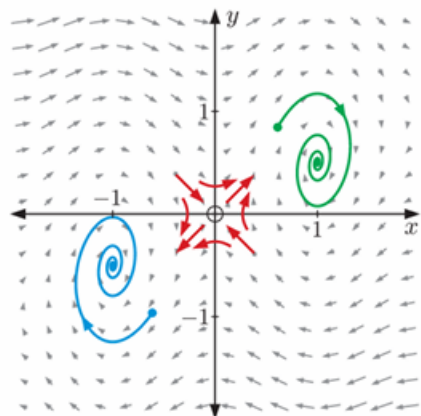
For example, for the coupled system in **Example 1**:

$$\frac{dy}{dt} = 0 \text{ whenever } \pi x = k\pi, \quad k \in \mathbb{Z}$$
$$\therefore x = k, \quad k \in \mathbb{Z}$$

$$\frac{dx}{dt} = 0 \text{ when } y - \frac{1}{2}x = 0$$

$\therefore$  for an equilibrium point,  $x = k, y = \frac{1}{2}k, k \in \mathbb{Z}$ .

On the phase portrait we observe two types of equilibrium point. There is a saddle point at  $(0, 0)$  and there are stable spirals at  $(-1, -\frac{1}{2})$  and  $(1, \frac{1}{2})$ .



(6)  $\det(A - \lambda \cdot I) = 0 \Leftrightarrow \det \begin{bmatrix} -0.05 - \lambda & 5 \\ -5 & -0.05 - \lambda \end{bmatrix} = 0$

$I =$  Identity Matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

main diagonal.

$$(-0.05 - \lambda) \cdot (-0.05 - \lambda) - 5 \cdot (-5) = 0$$

$$(-0.05 - \lambda)^2 + 25 = 0 \Leftrightarrow$$

$$(-0.05 - \lambda)^2 = -25$$

$$\sqrt{(-0.05 - \lambda)^2} = \sqrt{25i^2}$$

$$A - \lambda \cdot I = \begin{bmatrix} -0.05 & 5 \\ -5 & -0.05 \end{bmatrix} - \lambda \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.05 & 5 \\ -5 & -0.05 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A - \lambda \cdot I = \begin{bmatrix} -0.05 - \lambda & 5 \\ -5 & -0.05 - \lambda \end{bmatrix}^*$$

(6) (i)  $a \neq 0$ . (spiral)  
 (ii)  $a = -0.05$  in both cases

When we have a differential equation the value of  $a$  in complex eigen value determines if we have spiral inward motion (to the fixed point of the exercise O).

It won't move vertically, it will move in a spiral (if either  $a$  or  $b$  was zero, the speedboat would move in 2 different axis).

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \cdot d - b \cdot c$$

6. [Maximum mark: 15]

A model speedboat has its position, at time  $t$  seconds  $t \geq 0$ , defined by

$$\frac{dx}{dt} = 5y - 0.05x, \quad \frac{dy}{dt} = -5x - 0.05y,$$

where  $x$  metres is the distance east and  $y$  metres is the distance north of a fixed point O.

(a) Find the eigenvalues of  $A = \begin{pmatrix} -0.05 & 5 \\ -5 & -0.05 \end{pmatrix}$ , giving your answers in the form  $a + bi$ , where  $a \neq 0, b \neq 0$ . [4]

(b) (i) State what  $a \neq 0$  indicates about the path of the speedboat. [1]  
 (ii) State what the sign of  $a$  indicates about the path of the speedboat. [1]

At time  $t = 0$ , the speedboat has position  $(20, 0)$ .

(c) At time  $t = 0$ , find the value of

(i)  $\frac{dy}{dt}$ . [1]  
 (ii)  $\frac{dy}{dx}$ . [1]

(d) Use your answers to parts (b) and (c) to sketch the path of the model speedboat. [4]



$$-0.05 - \lambda = \pm 5i$$

$$-\lambda = 0.05 \pm 5i$$

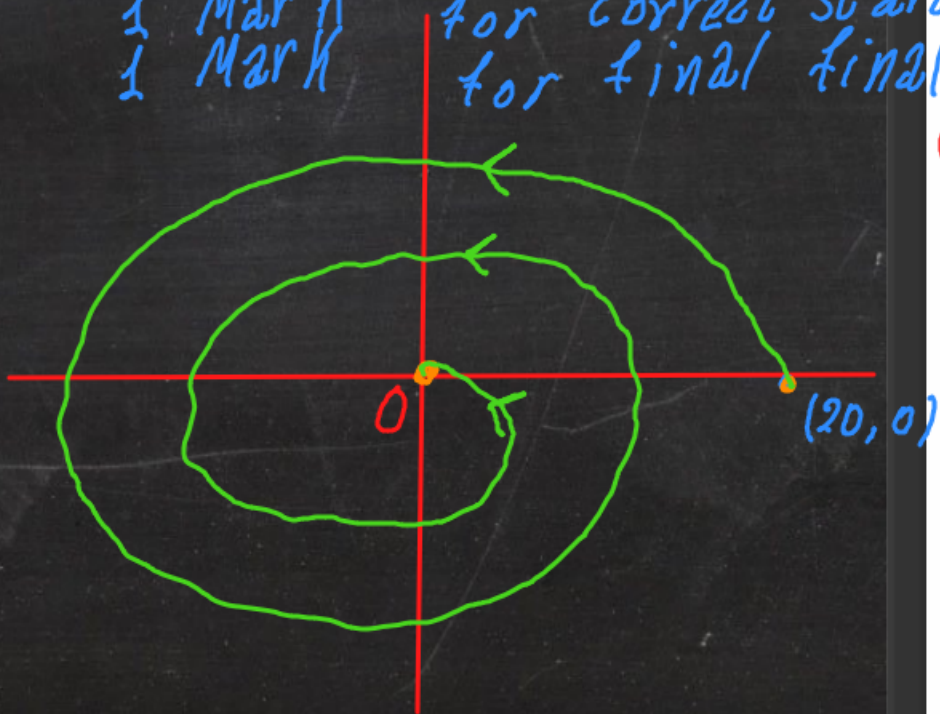
$$-\lambda_1 = 0.05 + 5i \quad \text{or} \quad -\lambda_2 = 0.05 - 5i$$

$$\lambda_1 = -0.05 - 5i \quad \text{or} \quad \lambda_2 = -0.05 + 5i$$

$a < 0$  so inward motion.  
 or an outward motion (away from the fixed point)

Paper II - 2023

- 1 Mark for spiral (curves)
- 1 Mark for correct arrows
- 1 Mark for correct starting position
- 1 Mark for final position



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where  $x$  metres is the distance east and  $y$  metres is the distance north of a fixed point  $O$ .

(a) Find the eigenvalues of  $A = \begin{pmatrix} -0.05 & 5 \\ -5 & -0.05 \end{pmatrix}$ , giving your answers in the form  $a + bi$ ,

where  $a \neq 0, b \neq 0$ .

[4]

(b) (i) State what  $a \neq 0$  indicates about the path of the speedboat.

(ii) State what the sign of  $a$  indicates about the path of the speedboat.

[2]

At time  $t = 0$ , the speedboat has position  $\begin{matrix} x & y \\ (20, & 0) \end{matrix}$ .

(c) At time  $t = 0$ , find the value of

(i)  $\frac{dy}{dt} = -5 \cdot 20 - 0.05 \cdot 0 = -100 \text{ ms}^{-1}$  (speed)

(ii)  $\frac{dx}{dt} = 5 \cdot 0 - 0.05 \cdot 20 = -1$

[5]

(d) Use your answers to parts (b) and (c) to sketch the path of the model speedboat.

[4]

$$\left( \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \right) = \frac{\cancel{dy} \cdot \cancel{dt}}{\cancel{dx} \cdot \cancel{dt}} = \frac{dy}{dx} = \frac{-100}{-1} = 100$$

7. [Maximum mark: 18]

## Paper II (2023)

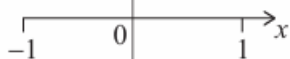
A trapezoid, Q, has vertices  $(0, -1)$ ,  $(0, -2)$ ,  $(\sin 15^\circ, -3 - \cos 15^\circ)$ ,  $(\sin 15^\circ, -1 - \cos 15^\circ)$  as shown.

a) Area of trapezoid:  $A_{tr} = \frac{1}{2} \cdot (\text{BASE} + \text{base}) \cdot \text{height}$

$$\text{height} = \sin 15^\circ$$

$$\text{base} = 1$$

$$\text{BASE} = 2$$



$$(0, -1)$$

$$(0, -2) \quad (\sin 15^\circ, -1 - \cos 15^\circ)$$

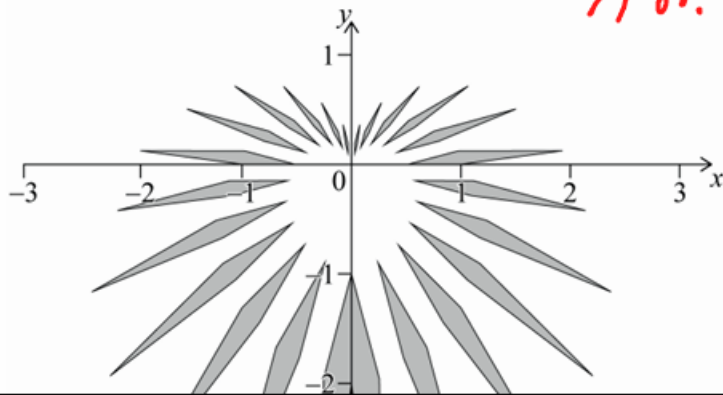
$$(\sin 15^\circ, -3 - \cos 15^\circ)$$

$$A_{tr} = \frac{1}{2} \cdot (2 + 1) \cdot \sin 15^\circ$$

(a) Show that the area of the trapezoid is  $\frac{3}{2} \sin 15^\circ$ .

[2]

A design is created with 24 elements. Each element is obtained by transforming the trapezoid Q. These elements are shaded in the following diagram such that the  $y$ -axis is a line of symmetry.

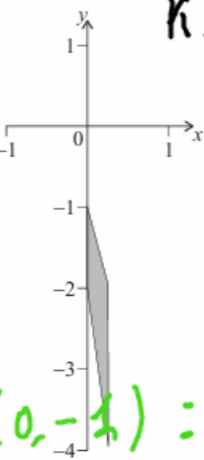


$$A_{tr} = 0.388 \text{ square units.}$$

A trapezoid, Q, has vertices  $(0, -1)$ ,  $(0, -2)$ ,  $(\sin 15^\circ, -3 - \cos 15^\circ)$ ,  $(\sin 15^\circ, -1 - \cos 15^\circ)$  as shown.

(b) (i) Substitute

$$M_6 = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$

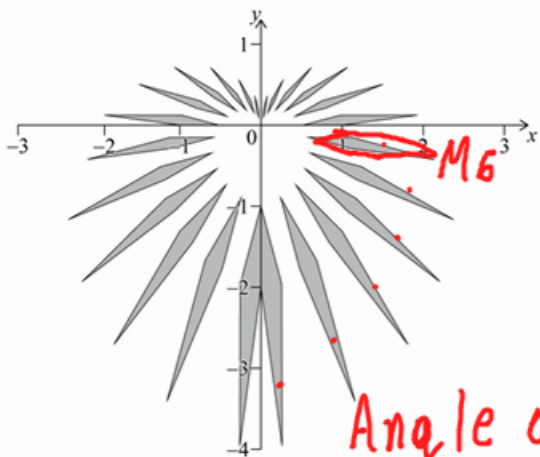


$$k=6 \Rightarrow M_6 = \begin{bmatrix} \frac{1}{2} \cos 90 & -\frac{1}{2} \sin 90 \\ \frac{1}{2} \sin 90 & \frac{1}{2} \cos 90 \end{bmatrix}$$

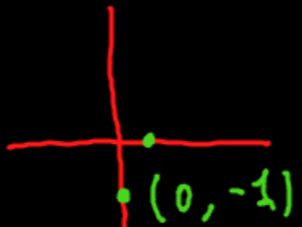
(ii) Image of  $(0, -1)$ :  $M_6 \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} =$

(a) Show that the area of the trapezoid is  $\frac{3}{2} \sin 15^\circ$ . [2]

A design is created with 24 elements. Each element is obtained by transforming the trapezoid Q. These elements are shaded in the following diagram such that the y-axis is a line of symmetry.



$$\begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \left(\frac{1}{2}, 0\right)$$



Angle of rotation: Just the angle  
 Scale Factor of enlargement:  
 only write the number

(Question 7 continued)

Chapter 15: Graph Theory  
 Chapter 14 - Haase = Affine Transformation

The transformation that produces each of the elements on the right side of the design can be represented by a matrix of the form

$$M_k = \begin{pmatrix} \left(1 - \frac{k}{12}\right) \cos(k \times 15^\circ) & -\left(1 - \frac{k}{12}\right) \sin(k \times 15^\circ) \\ \left(1 - \frac{k}{12}\right) \sin(k \times 15^\circ) & \left(1 - \frac{k}{12}\right) \cos(k \times 15^\circ) \end{pmatrix}$$

where  $k = 0, 1, 2, 3, \dots, 11$ .

- (b) (i) Find the matrix  $M_6$ . Give your answer in the form  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  where  $a, b, c, d \in \mathbb{Q}$ .
- (ii) Hence find the coordinates of the image of the vertex  $(0, -1)$  after it is transformed by the matrix  $M_6$ . [4]

The matrix  $M_k$  can be expressed as the product of a rotation matrix and an enlargement matrix.

- (c) Write down, in terms of  $k$ ,
  - (i) the rotation matrix;  $\rightarrow C_1 = \begin{bmatrix} \cos(k \cdot 15) & \sin(k \cdot 15) \\ \sin(k \cdot 15) & \cos(k \cdot 15) \end{bmatrix}$
  - (ii) the enlargement matrix;  $\rightarrow C_2 = \begin{bmatrix} 1 - \frac{k}{12} & 0 \\ 0 & 1 - \frac{k}{12} \end{bmatrix}$
  - (iii) the angle of the rotation;
  - (iv) the scale factor of the enlargement. [4]

check the inside  $k \cdot 15$

$$1 - \frac{k}{12}$$

that is in the front of every element:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \det A = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \underline{a \cdot d - b \cdot c}$$

7. [Maximum mark: 18]

A trapezoid, Q, has vertices  $(0, -1)$ ,  $(0, -2)$ ,  $(\sin 15^\circ, -3 - \cos 15^\circ)$ ,  $(\sin 15^\circ, -1 - \cos 15^\circ)$  as shown.

$$d) \det M_k = \det \begin{bmatrix} \left(1 - \frac{k}{12}\right) \cos(k \cdot 15^\circ) & -\left(1 - \frac{k}{12}\right) \sin(k \cdot 15^\circ) \\ \left(1 - \frac{k}{12}\right) \sin(k \cdot 15^\circ) & \left(1 - \frac{k}{12}\right) \cos(k \cdot 15^\circ) \end{bmatrix} =$$

$$\left(1 - \frac{k}{12}\right)^2 \cdot \cos^2(k \cdot 15^\circ) + \left(1 - \frac{k}{12}\right)^2 \cdot \sin^2(k \cdot 15^\circ) =$$

$$\left(1 - \frac{k}{12}\right)^2 \cdot \underbrace{[\cos^2(k \cdot 15^\circ) + \sin^2(k \cdot 15^\circ)]}_1 = \boxed{\left(1 - \frac{k}{12}\right)^2}$$

e) Total Area = 2 x Right Side

$$A_2 = \left(1 - \frac{2}{12}\right) \cdot \frac{3}{2} \sin 15^\circ$$

$$A_5 = \left(1 - \frac{5}{12}\right) \cdot \frac{3}{2} \sin 15^\circ$$

$$A_3 = \left(1 - \frac{3}{12}\right) \cdot \frac{3}{2} \sin 15^\circ$$

$$A_6 = \left(1 - \frac{6}{12}\right) \cdot \frac{3}{2} \sin 15^\circ$$

$$A_4 = \left(1 - \frac{4}{12}\right) \cdot \frac{3}{2} \sin 15^\circ$$

$$\dots$$

$$A_{right} = \frac{3}{2} \cdot \sin 15^\circ \cdot \left(1 + \frac{11^2 + 10^2 + 9^2 + 8^2 + 7^2 + 6^2 + 5^2 + 4^2 + 3^2 + 2^2 + 1^2}{12^2}\right)$$

$A_1 = 0.388$  square units.

$A_{total} = 2 \times 1.75 = 3.50$  square units

Talking: Mr Alex

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(Question 7 continued)

The transformation that produces each of the elements on the right side of the design can be represented by a matrix of the form

$$M_k = \begin{pmatrix} \left(1 - \frac{k}{12}\right) \cos(k \times 15^\circ) & -\left(1 - \frac{k}{12}\right) \sin(k \times 15^\circ) \\ \left(1 - \frac{k}{12}\right) \sin(k \times 15^\circ) & \left(1 - \frac{k}{12}\right) \cos(k \times 15^\circ) \end{pmatrix} \text{ careful!}$$

where  $k = 0, 1, 2, 3, \dots, 11$ .

- (b) (i) Find the matrix  $M_6$ . Give your answer in the form  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  where  $a, b, c, d \in \mathbb{Q}$ .  
 (ii) Hence find the coordinates of the image of the vertex  $(0, -1)$  after it is transformed by the matrix  $M_6$ . [4]

The matrix  $M_k$  can be expressed as the product of a rotation matrix and an enlargement matrix.

- (c) Write down, in terms of  $k$ ,  
 (i) the rotation matrix;  
 (ii) the enlargement matrix;  
 (iii) the angle of the rotation;  
 (iv) the scale factor of the enlargement. [4]  
 (d) Using your answer to part (c)(iv), or otherwise, find the determinant of the matrix  $M_k$  in terms of  $k$ . ✓ [2]  
 (e) Hence, or otherwise, find the total area of the elements in the **whole** design. ✓ [4]

Each element on the **left side** of the design can be obtained through a transformation of the trapezoid Q by applying the matrix  $N_k$ , where  $k = 0, 1, 2, 3, \dots, 11$ .

- (f) Write down the matrix  $N_0$  as a product of two matrices. [2]

Be very carefull we  
apply all  $k$  transformations  
before the matrix

(e) Hence, or otherwise, find the total area of the elements in the **whole** design. [4]

Each element on the **left side** of the design can be obtained through a transformation of the trapezoid  $Q$  by applying the matrix  $N_k$ , where  $k = 0, 1, 2, 3, \dots, 11$ .

(f) Write down the matrix  $N_k$  as a product of two matrices. [2]

$$N_k = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot M_k$$

square units = 3.50

$$\textcircled{1} \textcircled{a} m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9.45 - 8.73}{1958 - 1708} = 0.00288$$

$\textcircled{b} \textcircled{i}$  It is the rate of increase in the annual temperature.

$\textcircled{ii}$   $m \rightarrow$  temperature/years

$\textcircled{c}$  Use  $(1958, 9.45)$  to find the  $y$ -intercept  $c$ .

$$9.45 = 1958 \cdot 0.00288 + c \Rightarrow c = 3.81 \Rightarrow$$

$$y = 0.00288x + 3.81$$

$$\textcircled{d} y(2000) = 0.00288 \cdot 2,000 + 3.81 = 9.57$$

$$\textcircled{e} \textcircled{i} \text{ Linear Reg } (ax+b) \rightarrow a = 0.00256$$

$$b = 4.46$$

$$r = 0.861$$

$$\textcircled{ii} y = 0.00256x + 4.46$$

$$\textcircled{f} \text{ For } x=2000 \Rightarrow y = 0.00256 \cdot 2000 + 4.46$$

$$y = 9.58$$

marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 13]

The mean annual temperatures for Earth, recorded at fifty-year intervals, are shown in the table.

Year (x)	1708	1758	1808	1858	1908	1958	2008
Temperature °C (y)	8.73	9.22	9.10	9.12	9.13	9.45	9.76

Tami creates a linear model for this data by finding the equation of the straight line passing through the points with coordinates  $(1708, 8.73)$  and  $(1958, 9.45)$ .

- $x_1 \ y_1 \ x_2 \ y_2$
- $\checkmark$  (a) Calculate the gradient of the straight line that passes through these two points. [2]
  - $\checkmark$  (b) (i) Interpret the meaning of the gradient in the context of the question. [2]
  - $\checkmark$  (ii) State appropriate units for the gradient. [2]
  - $\checkmark$  (c) Find the equation of this line giving your answer in the form  $y = mx + c$ . [2]
  - $\checkmark$  (d) Use Tami's model to estimate the mean annual temperature in the year 2000. [2]

Thandizo uses linear regression to obtain a model for the data.

- $\checkmark$  (e) (i) Find the equation of the regression line  $y$  on  $x$ . [3]
- $\checkmark$  (ii) Find the value of  $r$ , the Pearson's product-moment correlation coefficient. [3]
- (f) Use Thandizo's model to estimate the mean annual temperature in the year 2000. [2]

$$a = \frac{y_{\max} - y_{\min}}{2} = \frac{18 - 4}{2} = 7 \Leftrightarrow \boxed{a = 7 \text{ m}}$$

$$d = \frac{y_{\max} + y_{\min}}{2} = \frac{18 + 4}{2} = \frac{22}{2} = 11 \Leftrightarrow \boxed{d = 11 \text{ m}}$$

$$\frac{T}{2} = 375 - 0 \Leftrightarrow T = 375 \times 2 \Leftrightarrow \boxed{T = 750 \text{ min}}$$

d)  $b = \text{angular velocity} - \text{speed (rad/sec)}$

$$b = \frac{360}{T} = \frac{360}{750} \Leftrightarrow \boxed{b = 0.48}$$

$$W(375) = 4 \Leftrightarrow 7 \cdot \cos(b \cdot 375) + 11 = 4 \Leftrightarrow \cos(375b) = \frac{4 - 11}{7} \Leftrightarrow \cos(375b) = -1$$

$$\cos(375b) = -1 \Leftrightarrow \cos(375b) = \cos(180) \Leftrightarrow \frac{375b}{375} = \frac{180}{375}$$

e) or way 2:  $W(t) = 6 \Leftrightarrow 7 \cos(0.48t) + 18 = 6$

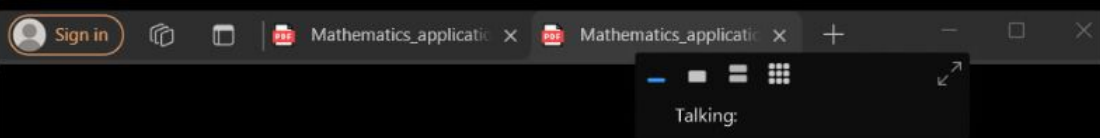
$$\text{GDC-SolveN}(7 \cos(0.48t) + 11 = 6) \Rightarrow t = 282 \text{ minutes}$$

$t = 4 \text{ h and } 42 \text{ minutes.}$

(Latest time:  $06:00 \text{ am} + 4 \text{ h } 42 \text{ min} = 10:42 \text{ am}$ )

f) From  $06:00 \text{ am} \rightarrow 10:42 \text{ am}$  she can enter and leave. After that she can't leave for a few hours.

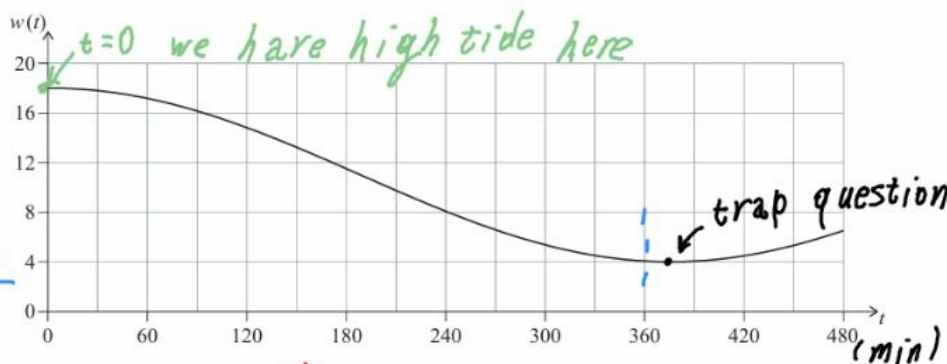
The second solution:  $489.6 - 282.4 = 207 \text{ minutes}$



2. [Maximum mark: 15]

The depth of water,  $w$  metres, in a particular harbour can be modelled by the function  $w(t) = a \cos(bt) + d$  where  $t$  is the length of time, in minutes, after 06:00.

On 20 January, the first high tide occurs at 06:00, at which time the depth of water is  $18 \text{ m}$ . The following low tide occurs at 12:15 when the depth of water is  $4 \text{ m}$ . This is shown in the diagram.



- Find the value of  $a$ . ✓ [2]
- Find the value of  $d$ . ✓ [2]
- Find the period of the function in minutes. ✓ [3]
- Find the value of  $b$ . ✓ [2]

Naomi is sailing to the harbour on the morning of 20 January. Boats can enter or leave the harbour only when the depth of water is at least  $6 \text{ m}$ .

- Find the latest time before 12:00, to the nearest minute, that Naomi can enter the harbour. [4]
- Find the length of time (in minutes) between 06:00 and 15:00 on 20 January during which Naomi cannot enter or leave the harbour. [2]

$$y_1 = 7 \cos(360t) + 18 / y_2 = 6$$

$$8.16 \times 60 = 489.6$$

interest

DU = doesn't use substances / TN = tests negative

$P(DU \cap TN) = P(DU) \cdot P(TN) = 0.921$   
(because we have independent events).

$P(A \cap B) = P(A) \cdot P(B)$

$(ii) (P(DU \cap TN))^2 = 0.921^2 = 0.849$

(because the athletes are independent of each other)  
(this is the total probability theorem)

$(c) P(\text{Incorrect}) = P(DU \cap TP) + P(U \cap TN) = P(DU) \cdot P(TP) + P(U) \cdot P(TN)$   
(all the events are independent so we multiply them).

$(ii) E(x) = n \cdot p = 1,300 \times 0.0362 = 47.06 = 47 \text{ athletes.}$

$(d) n = 20 / p = 0.02$  binomial probability.

c = cumulative (it adds up).

$P(x=0) = \binom{20}{0} \cdot 0.02^0 \cdot 0.98^{20} = 0.668.$

$(c) P(x > 2) \text{ or } P(x \geq 3) = 1 - P(x \leq 2) = 1 - (P(x=0) + P(x=1) + P(x=2))$

B-CDF (Lower: 3 upper: 20  $p=0.02$ )  $\Rightarrow P(x \geq 3) = 0.00707$

3. [Maximum mark: 17]

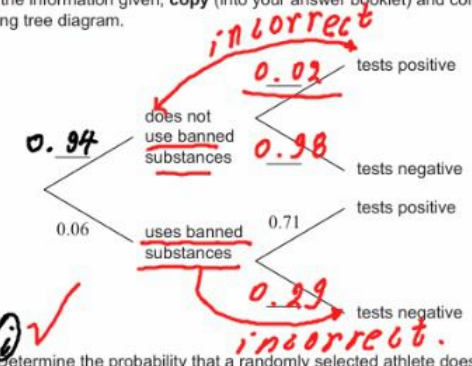
A large international sports tournament tests their athletes for banned substances. They interpret a positive test result as meaning that the athlete uses banned substances. A negative result means that they do not.

$P(U) \cdot P(U \cap TN) = 0.94 \cdot 0.02 + 0.06 \cdot 0.29 = 0.0362$   
The probability that an athlete uses banned substances is estimated to be 0.06.

If an athlete **uses** banned substances, the probability that they will test positive is 0.71.

If an athlete does **not use** banned substances, the probability that they will test negative is 0.98.

(a) Using the information given, **copy** (into your answer booklet) and complete the following tree diagram. [2]



(i) Determine the probability that a randomly selected athlete does not use banned substances and tests negative.

(ii) If two athletes are selected at random, calculate the probability that both athletes do not use banned substances and both test negative. [4]

(c) (i) Calculate the probability that a randomly selected athlete will receive an **incorrect** test result.

(ii) A random sample of 1300 athletes at the tournament are selected for testing. Calculate the expected number of athletes in the sample that will receive an incorrect test result. [5]

Team X are competing in the tournament. There are 20 athletes in this team. It is known that none of the athletes in Team X use **banned substances**.

(d) Calculate the probability that none of the athletes in Team X will test positive. [4]

(e) Determine the probability that more than 2 athletes in Team X will test positive. [2]