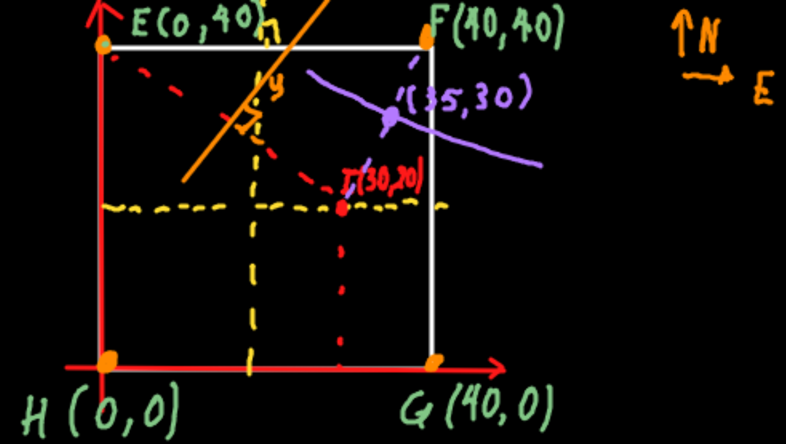
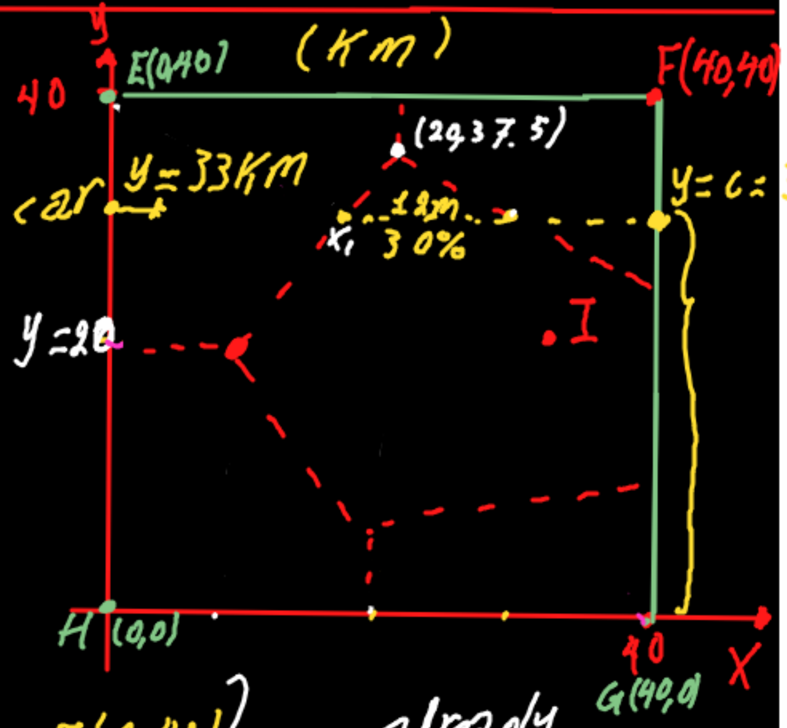
(b) The points closest between a pair of cities, are always the points that lie on the perpendicular bisector (isosceles triangle)

(c) $IE: y_{b_{IE}} = \frac{3}{2}x + \frac{15}{2}$. $m_{IF} = \frac{y_F - y_I}{x_F - x_I} = 2 \Rightarrow m_{bi_{IF}} = -\frac{1}{2}$ $(m_{IF} \cdot m_{bi_{IF}} = -1)$. M_{IF}: $x_M = \frac{x_I + x_F}{2} = 35$, $y_M = \frac{y_I + y_F}{2} = 30$

$y_{Bi_{IF}} = -\frac{1}{2}x + c \Rightarrow 30 = -0.5 \cdot 35 + c \Rightarrow c = 47.5 \Rightarrow$ $\boxed{y_{Bi_{IF}} = -\frac{1}{2}x + \frac{95}{2}}$





2. [Maximum mark: 27]

This question is about a metropolitan area council planning a new town and the location of a new toxic waste dump.

A metropolitan area in a country is modelled as a square. The area has four towns, located at the corners of the square. All units are in kilometres with the x-coordinate representing the distance east and the y-coordinate representing the distance north from the origin at $(0, 0)$.

- Edison is modelled as being positioned at $E(0, 40)$.
- Fermitown is modelled as being positioned at $F(40, 40)$.
- Gaussville is modelled as being positioned at $G(40, 0)$.
- Hamilton is modelled as being positioned at $H(0, 0)$.

(a) The model assumes that each town is positioned at a single point. Describe possible circumstances in which this modelling assumption is reasonable. [1]

Each coordinate represent just the center of the town

Each town compared to the distances between each city is small

(b) Sketch a Voronoi diagram showing the regions within the metropolitan area that are closest to each town. [1]

The metropolitan area council decides to build a new town called Isaacopolis located at $I(30, 20)$.

A new Voronoi diagram is to be created to include Isaacopolis. The equation of the perpendicular bisector of [IE] is $y = \frac{3}{2}x + \frac{15}{2}$.

(c) (i) Find the equation of the perpendicular bisector of [IF]. [4]

(ii) Given that the coordinates of one vertex of the new Voronoi diagram are $(20, 37.5)$, find the coordinates of the other two vertices within the metropolitan area. [4]

(iii) Sketch this new Voronoi diagram showing the regions within the metropolitan area which are closest to each town. (the doted lines) [2]

The metropolitan area is divided into districts based on the Voronoi regions found in part (c).

(d) A car departs from a point due north of Hamilton. It travels due east at constant speed to a destination point due North of Gaussville. It passes through the Edison, Isaacopolis and Fermitown districts. The car spends 30% of the travel time in the Isaacopolis district.

Find the distance between Gaussville and the car's destination point. [4]

(c)(ii) When 2 or more bisectors meet, at this point we have a vertex

$\begin{cases} y = 20 \ (EH) \\ y = \frac{3}{2}x + \frac{15}{2} \ (IE) \end{cases}$

$\begin{cases} y_{br_{IG}} = \frac{1}{2}x - \frac{15}{2} \\ x = 20 \ (Bi \text{ of } KG) \end{cases} \Rightarrow (20, 2.5)$

$\begin{cases} y = -\frac{3}{2}x + \frac{65}{2} \\ x = 20 \end{cases}$

E(0,40), H(0,0): K is already on the perpendicular of EF.

(d) 30% of 40 Km = 12Km

c = 33Km from G

$\begin{cases} y_{IF} = -\frac{1}{2}x + \frac{95}{2} \ (S1) \\ y = c \end{cases}$ $\quad$ $\begin{cases} y = \frac{3}{2}x + \frac{15}{2} \ (S2) \\ y = c \end{cases}$

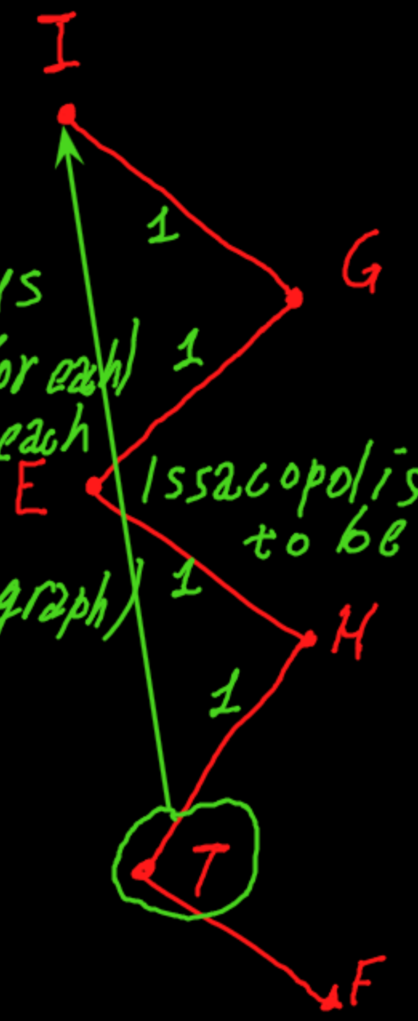
$2c = -x_2 + 95$ $\quad$ $x_1 = \frac{2c - 15}{3}$

$x_2 - x_1 = 12 \Rightarrow 2c - 95 - \frac{2c - 15}{3} = 12$

We have to take into account the geometry of a region, if you put in inside you create a cyclical region if you put it closer to a city but at the edges you can have less harmful results

(May 2019 / Paper III / Time Zone 1)

(f) (i)



Issacopolis is last to be polluted

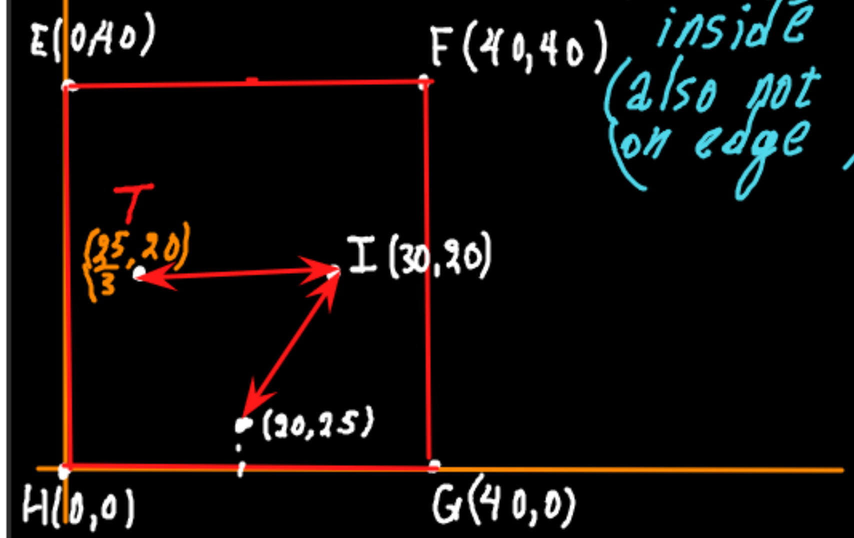
(ii) 4 days (1 day for each) (1 day for each Edge in the graph)

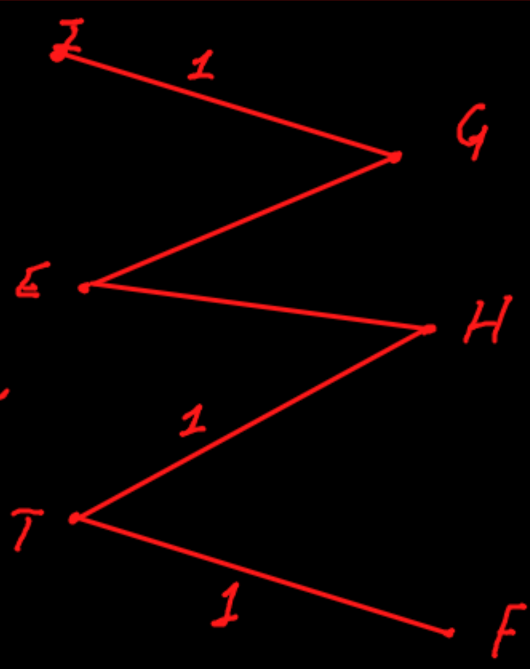
(iii) E : 2, F : 1, G : 2, H : 2, I : 1, T : 2

Start from I and end at F.

Fermitown, Isaacopolis, Edison, Gaussville, Hamilton, Toxic

(e) (i) The vertices inside the metropolitan area, are furthest from each corner, therefore it has to be one of the 3 vertices inside (also not on edge)





(Question 2 continued)

(population densities)

A toxic waste dump needs to be located within the metropolitan area. The council wants to locate it as far as possible from the nearest town.

(e) (i) Find the location of the toxic waste dump, given that this location is not on the edge of the metropolitan area. [4]

(ii) Make one possible criticism of the council's choice of location. [1]

(f) The toxic waste dump, T, is connected to the towns via a system of sewers.

The connections are represented in the following matrix, $\boldsymbol{M}$, where the order of rows and columns is (E, F, G, H, I, T).

(Graph)

$$
\boldsymbol{M} = \begin{array}{c} \\ E \\ F \\ G \\ H \\ I \\ T \end{array}
\begin{array}{c} \begin{array}{cccccc} E & F & G & H & I & T \end{array} \\
\begin{pmatrix}
1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1
\end{pmatrix} \end{array}
$$

sewer, polluted

A leak occurs from the toxic waste dump and travels through the sewers. The pollution takes one day to travel between locations that are directly connected.

The digit 1 in $\boldsymbol{M}$ represents a direct connection. The values of 1 in the leading diagonal of $\boldsymbol{M}$ mean that once a location is polluted it will stay polluted.

not care about diagonal when sketching the graph.

(i) Find which town is last to be polluted. Justify your answer. [3]

(ii) Write down the number of days it takes for the pollution to reach the last town. [1]

(iii) A sewer inspector needs to plan the shortest possible route through each of the connections between different locations. Determine an appropriate start point and an appropriate end point of the inspection route.

Note that the fact that each location is connected to itself does not correspond to a sewer that needs to be inspected. [2]

(a) (i) $\frac{dV}{dt} = \frac{dV}{ds} \cdot \frac{ds}{dt} = \frac{dV}{ds} \cdot V = V \cdot \boxed{\frac{dV}{ds}}$

The rate of change of displacement is velocity

Separable Equations

$V \cdot \frac{dV}{\cancel{ds}} \cdot \cancel{ds} = g \cdot ds \Leftrightarrow \int V\,dV = \int g\,ds \Leftrightarrow \frac{V^2}{2} + c = g \int 1\,ds$

$\frac{V^2}{2} + c = gs \Leftrightarrow \frac{V^2}{2} = g \cdot s - c \Leftrightarrow \cdot 2$

November 2023 (Paper 3)

$\boxed{\frac{dV}{dt} = V'(t)}$

$\boxed{\frac{ds}{dt} = s'(t) = V(t)}$

$\begin{cases} \text{displacement} = \int_{t_1}^{t_2} V(t)\,dt \\ s' = V(t) \end{cases}$

$\frac{ds}{dt} = V(t)$

$\boxed{(c = -50) \Rightarrow V = \sqrt{2gs + 100}}$

(iii) $V = 330 \Leftrightarrow$

SolveN$(330 = \sqrt{2 \cdot 9.8 s + 100}) \Rightarrow$

$S = 5{,}550\,m < 40{,}000\,m$

(5551.020408)

Method 2: $S = 40{,}000\,m$

he will reach the speed of sound.

1. [Maximum mark: 28]

This question uses differential equations to model the maximum velocity of a skydiver in free fall.

In 2012, Felix Baumgartner jumped from a height of $40\,000\,\text{m}$. He was attempting to travel at the speed of sound, $330\,\text{m s}^{-1}$, whilst free-falling to the Earth.

Before making his attempt, Felix used mathematical models to check how realistic his attempt would be. The simplest model he used suggests that

$$\frac{dv}{dt} = g$$

where $v\,\text{m s}^{-1}$ is Felix's velocity and $g\,\text{m s}^{-2}$ is the acceleration due to gravity. The time since he began to free-fall is t seconds and the displacement from his initial position is s metres.

Throughout this question, the direction towards the centre of the Earth is taken to be positive and v is a positive quantity.

When $s = 0$, it is given that Felix jumps with an initial velocity $v = 10$. ms^{-1}

(a) (i) Use the chain rule to show that $\frac{dv}{dt} = v\frac{dv}{ds}$. [1]

(ii) Assuming that g is a constant, solve the differential equation $v\frac{dv}{ds} = g$ to find v as a function of s. [4]

(iii) Using $g = 9.8$, determine whether the model predicts that Felix will succeed in travelling at the speed of sound at some point before $s = 40\,000$. Justify your answer. [3]

(b) To test the model

$$\frac{dv}{dt} = g\,,$$

Felix conducted a trial jump from a lower height, and data for v against t was found.

(i) If the model is correct, describe the shape of the gr [2]

$\left.\begin{matrix} dV \\ ds \end{matrix}\right\}$ combined they give you p

$\frac{dV}{dt} = a(t) \,/\, \frac{ds}{dt} = V(t)$

$\frac{d^{(2)}s}{dt} = \left(\frac{ds}{dt}\right)' = \frac{dV}{dt} = a$

$\cancel{2} \cdot \frac{V^2}{\cancel{2}} = 2 \cdot gs - 2c \Leftrightarrow$

$\sqrt{V^{\cancel{2}}} = \sqrt{2gs - 2c} \Leftrightarrow$

$|V| = \sqrt{2gs - 2c} \Leftrightarrow$

$V = \cancel{\pm}\sqrt{2gs - 2c}$ (according to the question $V > 0$)

for $s = 0$ / $V = 10 \Rightarrow$

SolveN$(10 = \sqrt{-2c}) \Rightarrow$

November 2023 (Paper 3)

(b) To test the model

$$\boxed{\frac{dv}{dt} = g ,}$$

Felix conducted a trial jump from a lower height, and data for v against t was found.

(i) If the model is correct, describe the shape of the graph of v against t. [2]

(This question continues on the following page)

$$\cancel{d}t \cdot \frac{dV}{\cancel{dt}} = g \cdot dt$$

$$\int 1 \, dV = \int g \cdot dt \Leftrightarrow \boxed{V = gt}$$

linear correlation, so

we have a straight line.

$\frac{dV}{dt} = g - KV^2 \Leftrightarrow V\cdot\frac{dV}{ds} = g - KV^2 \Leftrightarrow \cancel{ds}\cdot V\cdot\frac{dV}{\cancel{ds}} = (g - KV^2)ds \Leftrightarrow \frac{V}{g-KV^2}\cdot dV = \frac{\cancel{(g-KV^2)}\cdot ds}{\cancel{g-KV^2}} \Leftrightarrow$

$\int \frac{V}{g-KV^2}\, dV = \int 1\, ds \Leftrightarrow \int \frac{-1}{2K}\cdot\frac{1}{u}\, du = \int 1\, ds \Leftrightarrow \frac{-1}{2K}\cdot\int\frac{1}{u}\, du = \int 1\, ds \Leftrightarrow \frac{-1}{2K}\cdot\ln|u| = s + c \Leftrightarrow$

November 2023 (Paper 3)

Technique of Integration

Let $g - KV^2 = u \Rightarrow$

$d(g - KV^2) = du \Rightarrow$

$\frac{\cancel{-2K}V\,dV}{\cancel{-2K}} = \frac{du}{-2K} \Rightarrow$

$V\,dV = \frac{1}{-2K}\,du$

$\ln(g - KV^2) = -2K(s+c)$ *

When $s = 0$ / $V = 10 \Rightarrow$

$\ln(g - 100K) = -2Kc \Leftrightarrow$

$e^{\cancel{\ln}(g-100K)} = e^{-2Kc} \Leftrightarrow$

$\boxed{g - 100K = e^{-2Kc}}$

(Question 1 continued)

$\boxed{** \; V = \sqrt{\frac{g - (g - 100K)\cdot e^{-2Ks}}{K}}}$

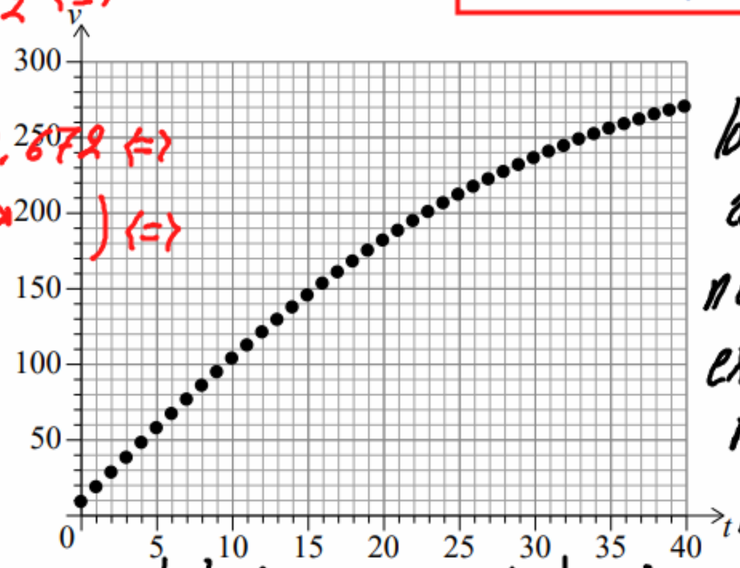
Felix's data are plotted on the following graph.

$\frac{dV}{dt} = 9.672 \Leftrightarrow$

$9.8 - K\cdot 40^2 = 9.672 \Leftrightarrow$

GDC SolveN( ) $\Leftrightarrow$

$\boxed{K = 8\cdot 10^{-5}}$



The model is not valid since, the graph that represents the real values the experiment do not illustrate a straight line but a curve.

(ii) Use the plot to comment on the validity of the model in part (a). [1]

(c) An improved model considers air resistance, using

$$\frac{dv}{dt} = g - kv^2$$

where k is a positive constant. You are reminded that initially $s = 0$ and $v = 10$.

(i) By using $\frac{dv}{dt} = v\frac{dv}{ds}$, solve the differential equation to find v in terms of s, g and k. You may assume that $g - kv^2 > 0$. [5]

Felix uses the graph of v against t shown in part (b) to estimate the value of k.

(ii) The gradient is estimated to be 9.672 when $v = 40$. Taking g to be 9.8, use this information to show that Felix found that $k = 8\times 10^{-5}$. [2]

$\cancel{-2K}\cdot\frac{\cancel{-1}}{\cancel{2K}}\cdot\ln|u| = -2Ks - 2Kc$

$g - KV^2 > 0$ according to the exercise

* $\ln(g - KV^2) = -2Ks - 2Kc \Leftrightarrow$

$e^{\cancel{\ln}(g-KV^2)} = e^{-2K(s+c)} \Leftrightarrow$

$g - KV^2 = e^{-2K(s+c)} \Leftrightarrow$

$\frac{\cancel{K}V^2}{\cancel{K}} = \frac{g - e^{-2K(s+c)}}{K} \Leftrightarrow$

$\sqrt{V^{\cancel{2}}} = \sqrt{\frac{g - e^{-2K(s+c)}}{K}} \Leftrightarrow$

$\boxed{** \; V = \sqrt{\frac{g - e^{-2K(s+c)}}{K}}}$

November 2023 (Paper III)

(iii) Hence, find the value of v predicted by this model, as s tends to infinity. [2]

(iv) Find the upper bound for the velocity according to this model, given that $0 < s \leq 40000$. Give your answer to four significant figures. [2]

(This question continues on the following page)

iii

$$V = \sqrt{\frac{g - (g - 100K)\cdot \cancel{e^{-2Ks}}}{K}}$$

$$\left(\text{If } s \to +\infty \quad e^{-2Ks} \to e^{-\infty} \to \cancel{\frac{1}{e^{+\infty}}} \to 0 \right)$$

so when $\left.\begin{array}{l} s \to +\infty \\ e^{-2Ks} \to 0 \end{array}\right\} \Rightarrow V_{max} = \sqrt{\frac{g}{K}} \Rightarrow \boxed{V_{max} = 350\ m/s}$

iv

$$V = \sqrt{\frac{9.8 - (9.8 - 100\cdot 8\cdot 10^{-5})\cdot e^{-2\cdot 8\cdot 10^{-5}\cdot 40{,}000}}{8\cdot 10^{-5}}} = 349.7\ ms^{-1}$$

Turn over

upper limit occurs when $s = 40{,}000$

November 2023 (Paper III)

ⓓ $X_{n+1} = X_n + h$

$S_{n+1} = S_n + 4{,}000$ (Euler Method)

(independent value)

$V_{n+1} = V_n + \overset{h}{4{,}000} \cdot \frac{dV}{ds}$ ⟺ ugly thing

$V_{n+1} = V_n + 4{,}000 \cdot \left(\frac{3.98 \cdot 10^{14}}{V_n(6.41 \times 10^6 - s)^2} - (8 \cdot 10^{-5}) \cdot V_n\right)$

If $\begin{cases} V_0 = 10 \\ s = 0 \end{cases} \Rightarrow V_{10} =$

$V_{10} = V_9 + \ldots$

$V_{10} = V_8 + 2 \ldots$

$V_{10} = V_7 + 3 \ldots$ u.t.

$V_{10} = V_6 + 4 \ldots$

$V_{10} = V_4 + 6 \ldots$

$V_{10} = V_0 + \cancel{10} \cdot 4{,}000 \cdot \left(\frac{3.98 \cdot 10^{14}}{\cancel{V_0}(6.41 \times 10^6 - s)^2} - (8 \cdot 10^{-5}) \cdot \cancel{V_0}\right)$ ⟺ $V_{10} = 10 + 4{,}000 \left(\frac{3.98 \cdot 10^{14}}{6.41^2 \cdot \cancel{100}} - 8 \cdot 10^{-5}\right)$

The assumption that the value of g is constant is not correct. It can be shown that

$$\boxed{g = \frac{3.98 \times 10^{14}}{(6.41 \times 10^6 - s)^2}.}$$

$\boxed{y_n = y_{n-1} + h \cdot \frac{dy}{dt}}$ derrivative

Hence, the new model is given by

$$\boxed{v\frac{dv}{ds} = \frac{3.98 \times 10^{14}}{(6.41 \times 10^6 - s)^2} - (8 \times 10^{-5})v^2.}$$

$\frac{dV}{ds} = \frac{3.98 \cdot 10^{14}}{V \cdot (6.41 \cdot 10^6 - s)^2} - \frac{(8 \times 10^{-5})V^{\cancel{2}}}{\cancel{V}}$ ⟺

When $s = 0$, it is known that $v = 10$.

(d) Use Euler's method with a step length of 4000 to estimate the value of v when $s = 40\,000$. [4]

(e) After Felix completed his record-breaking jump, he found that the answer from part (d) was not supported by data collected during the jump.

(awalys at Euler method)

(i) Suggest **one** improvement to the use of Euler's method which might increase the accuracy of the prediction of the model. [1]

(ii) Suggest **one** factor **not** explicitly considered by the model in part (d) which might lead to a difference between the model's prediction and the data collected. [1]

ⓘ Smaller step length ← More repetitions !!!
(Runge-Kutta Method)

ⓘⓘ Wind Method (Weather-Circumstances).

$V_{10} = 10 + 4000\left(\frac{3.98}{6.41^2} - 8 \cdot 10^{-3}\right)$ ⟺ $V_{10} = 361$ (360.658)